

# VECTOR CALCULUS

## TOPIC I: SETS AND FUNCTIONS

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### 1. SETS AND ELEMENTS

A *set* is a collection of *elements*. The elements of a set are sometimes called *members* or *points*. We assume that we can distinguish between different elements, and that we can determine whether or not a given element is in a given set.

The relationship of two elements  $a$  and  $b$  being the same is *equality* and is denoted  $a = b$ . The negation of this relation is denoted  $a \neq b$ , that is,  $a \neq b$  means that it is not the case that  $a = b$ .

The relationship of an element  $a$  being a member of a set  $A$  is *containment* and is denoted  $a \in A$ . The negation of this relation is denoted  $b \notin A$ , that is,  $b \notin A$  means that it is not the case that  $b \in A$ .

A set is determined by the elements it contains. That is, two sets are considered equal if and only if they contain the same elements. We use the symbols “ $\Rightarrow$ ” to mean “implies”, and “ $\Leftrightarrow$ ” to mean “if and only if”. Then

$$A = B \quad \Leftrightarrow \quad (a \in A \Leftrightarrow a \in B);$$

in English, “ $A$  equals  $B$  if and only if ( $a$  is in  $A$  if and only if  $b$  is in  $B$ )”.

We may describe a set by listing its members; such lists are surrounded by braces. For example the set of the first five prime integers is  $\{2, 3, 5, 7, 11\}$ . If a pattern is clear, we may use dots to indicate an infinite set; for example, to label the set of all prime numbers as  $P$ , we may write  $P = \{2, 3, 5, 7, 11, 13, \dots\}$ . The order of elements in a list is irrelevant in determining a set, for example,  $\{5, 3, 7, 11, 2\} = \{2, 3, 5, 7, 11\}$ . Also, there is no such thing as the “multiplicity” of an element in a set, for example  $\{1, 3, 2, 2, 1\} = \{1, 2, 3\}$ .

### 2. SUBSETS

If  $A$  and  $B$  are sets and all of the elements in  $A$  are also contained in  $B$ , we say that  $A$  is a *subset* of  $B$  or that  $A$  is *contained* in  $B$  and write  $A \subset B$ :

$$A \subset B \quad \Leftrightarrow \quad (a \in A \Rightarrow a \in B);$$

in English, “ $A$  is contained in  $B$  if and only if ( $a$  is in  $A$  implies  $a$  is in  $B$ )”. Every set is a subset of itself. We say that  $A$  is a *proper subset* of  $B$  is  $A \subset B$  but  $A \neq B$ .

It follows immediately from the definition of subset that

$$A = B \quad \Leftrightarrow \quad (A \subset B \text{ and } B \subset A);$$

in English, “ $A$  equals  $B$  if and only if ( $A$  is a subset of  $B$  and  $B$  is a subset of  $A$ )”.

A set containing no elements is called the *empty set* and is denoted  $\emptyset$ . Since a set is determined by its elements, there is only one empty set. Note that the empty set is a subset of any set.

### 3. SET OPERATIONS

We may construct new sets as subsets of existing sets by specifying properties. Specifically, we may have a proposition  $p(x)$  which is true for some elements  $x$  in a set  $X$  and not true for others. Then we may construct the set

$$\{x \in X \mid p(x) \text{ is true}\};$$

this is read “the set of  $x$  in  $X$  such that  $p(x)$ ”. The construction of this set is called *specification*. For example, if we let  $\mathbb{Z}$  be the set of integers, the set  $P$  of all prime numbers could be specified as  $P = \{n \in \mathbb{Z} \mid n \text{ is prime}\}$ .

Let  $A$  and  $B$  be subsets of some “universal set”  $U$  and define the following set operations:

$$\begin{aligned} \text{Union:} \quad & A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\} \\ \text{Intersection:} \quad & A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\} \\ \text{Complement:} \quad & A \setminus B = \{x \in U \mid x \in A \text{ and } x \notin B\} \end{aligned}$$

The pictures which correspond to these operations are called *Venn diagrams*.

**Example 1.** Let  $A = \{1, 3, 5, 7, 9\}$ ,  $B = \{1, 2, 3, 4, 5\}$ . Then  $A \cap B = \{1, 3, 5\}$ ,  $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$ ,  $A \setminus B = \{7, 9\}$ , and  $B \setminus A = \{2, 4\}$ .  $\square$

**Example 2.** Let  $A$  and  $B$  be two distinct nonparallel lines in a plane. We may consider  $A$  and  $B$  as sets of points. Their intersection is a set containing a single point, their union is a set consisting of all points on crossing lines, and the complement of  $A$  with respect to  $B$  is  $A$  minus the point of intersection.  $\square$

If  $A \cap B = \emptyset$ , we say that  $A$  and  $B$  are *disjoint*.

The following properties are sometimes useful.

- $A = A \cup A = A \cap A$
- $\emptyset \cap A = \emptyset$  and  $\emptyset \cup A = A$
- $A \subset B \Leftrightarrow A \cap B = A$
- $A \subset B \Leftrightarrow A \cup B = B$

The following properties state that union and intersection are commutative and associative operations, and that they distribute over each other. These properties are intuitively clear via Venn diagrams.

- $A \cap B = B \cap A$
- $A \cup B = B \cup A$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
- $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Since  $(A \cap B) \cap C = A \cap (B \cap C)$ , parentheses are useless and we write  $A \cap B \cap C$ . This extends to four sets, five sets, and so on. Similar remarks apply to unions.

The following properties of complement are known as *DeMorgan's Laws*. You should draw Venn diagrams of these situations to convince yourself that these properties are true.

- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

## 4. CARTESIAN PRODUCT

Let  $a$  and  $b$  be elements. The *ordered pair* with first coordinate  $a$  and second coordinate  $b$  consists of these two elements in the specified order. We denote this ordered pair by  $(a, b)$  and declare that it has the following “defining property”:

$$(a, b) = (c, d) \iff (a = c \text{ and } b = d).$$

The ordered pair  $(a, a)$  is allowed, and  $(a, b) = (b, a) \iff a = b$ .

The *cartesian product* of the sets  $A$  and  $B$  is denoted  $A \times B$  and is defined to be the set of all ordered pairs whose first coordinate is in  $A$  and whose second coordinate is in  $B$ :

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

**Example 3.** Let  $A = \{1, 3, 5\}$  and let  $B = \{1, 4\}$ . Then

$$A \times B = \{(1, 1), (1, 4), (3, 1), (3, 4), (5, 1), (5, 4)\}.$$

In particular, this set contains 6 elements.  $\square$

In general, if  $A$  contains  $m$  elements and  $B$  contains  $n$  elements, where  $m$  and  $n$  are natural numbers, then  $A \times B$  contains  $mn$  elements. Consider the case where  $A = B$ ; then  $A \times A$  contains  $m^2$  elements. We sometimes write  $A^2$  to mean  $A \times A$ .

We have the following properties of cartesian products:

- $(A \cup B) \times C = (A \times C) \cup (B \times C)$ ;
- $(A \cap B) \times C = (A \times C) \cap (B \times C)$ ;
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$ ;
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$ ;
- $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$ .

## 5. NUMBERS

The following familiar sets of numbers have standard names:

$$\text{Natural Numbers: } \mathbb{N} = \{0, 1, 2, 3, \dots\}$$

$$\text{Integers: } \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

$$\text{Rational Numbers: } \mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\right\}$$

$$\text{Real Numbers: } \mathbb{R} = \{\text{numbers given by decimal expansions}\}$$

$$\text{Complex Numbers: } \mathbb{C} = \{a + ib \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}$$

We have  $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ .

The following standard notation gives subsets of the real numbers, called *intervals*:

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$  (closed)
- $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$  (open)
- $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
- $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
- $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$  (closed)
- $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$  (open)
- $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$  (closed)
- $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$  (open)

## 6. FUNCTIONS

Let  $A$  and  $B$  be sets. A *function* from a set  $A$  to a set  $B$  is an assignment of every element in  $A$  to a unique element in  $B$ . Alternatively, a function is a method of sending each element of  $A$  to an element of  $B$ .

Let  $f$  be a function from  $A$  to  $B$ . If  $a \in A$ , the element of  $B$  to which  $a$  is assigned by  $f$  is denoted  $f(a)$ ; in other words, the place in  $B$  to which  $a$  is sent by  $f$  is denoted  $f(a)$ . We declare that a function must satisfy the following “defining property”:

for every  $a \in A$  there exists a unique  $b \in B$  such that  $f(a) = b$ .

If  $f$  is a function from  $A$  to  $B$ , this fact is denoted

$$f : A \rightarrow B.$$

We say that  $f$  *maps*  $A$  *into*  $B$ , and that  $f$  is a function *on*  $A$ . For this reason, functions are sometimes called *maps* or *mappings*. If  $f(a) = b$ , we say that  $a$  is *mapped to*  $b$  by  $f$ . We may indicate this by writing  $a \mapsto b$ .

Two functions  $f : A \rightarrow B$  and  $g : A \rightarrow B$  are considered *equal* if they act the same way on every element of  $A$ :

$$f = g \quad \Leftrightarrow \quad (a \in A \Rightarrow f(a) = g(a)).$$

Thus to show that two functions  $f$  and  $g$  are equal, select an arbitrary element  $a \in A$  and show that  $f(a) = g(a)$ .

If  $A$  is sufficiently small, we may explicitly describe the function by listing the elements of  $A$  and where they go; for example, if  $A = \{1, 2, 3\}$  and  $B = \mathbb{R}$ , a perfectly good function is described by  $\{1 \mapsto 23.432, 2 \mapsto \pi, 3 \mapsto \sqrt{593}\}$ .

However, if  $A$  is large, the functions which are easiest to understand are those which are specified by some *rule* or *algorithm*. The common functions of single variable calculus are of this nature.

**Example 4.** The following can be functions from  $\mathbb{R}$  into  $\mathbb{R}$ :

- $f(x) = 0$ ;
- $f(x) = x$ ;
- $f(x) = x^3 + 3x + 17$ .

The following can be functions from the set of positive real numbers into  $\mathbb{R}$ :

- $f(x) = \frac{1}{x}$ ;
- $f(x) = \sqrt{x}$ .

Note that  $\frac{1}{x}$  is not a function from  $\mathbb{R}$  into  $\mathbb{R}$ , because it is not defined at  $x = 0$ .  $\square$

Some functions are constructed from existing functions by specifying cases.

**Example 5.** Let  $\mathbb{R}$  be the set of real numbers. Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} x^2 + 2 & \text{if } x < 0; \\ x^3 - 1 & \text{if } x \geq 0. \end{cases}$$

Then, for example,  $f(-2) = (-2)^2 + 2 = 6$  and  $f(2) = 2^3 - 1 = 7$ .  $\square$

**Example 6.** Let  $X$  be a set and let  $A \subset X$ . The *characteristic function* of  $A$  in  $X$  is a function  $\chi_A : X \rightarrow \{0, 1\}$  defined by

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A; \\ 1 & \text{if } x \in A. \end{cases}$$

## 7. IMAGES AND PREIMAGES

If  $f : A \rightarrow B$ , the set  $A$  is called the *domain* of the function and the set  $B$  is called the *codomain*. We often think of a function as taking the domain  $A$  and placing it in the codomain  $B$ . However, when it does so, we must realize that more than one element of  $A$  can be sent to a given element in  $B$ , and that there may be some elements in  $B$  to which no elements of  $A$  are sent.

If  $a \in A$ , the *image* of  $a$  under  $f$  is  $f(a)$ .

If  $b \in B$ , the *preimage* of  $b$  is a subset of  $A$  given by

$$f^{-1}(b) = \{a \in A \mid f(a) = b\}.$$

If  $C \subset A$ , we define the *image* of  $C$  under  $f$  to be the set

$$f(C) = \{b \in B \mid f(a) = b \text{ for some } a \in C\}.$$

The image of the domain is called the *range* of the function.

If  $D \subset B$ , we define the *preimage* of  $D$  under  $f$  to be the set

$$f^{-1}(D) = \{a \in A \mid f(a) \in D\}.$$

Notice that  $f^{-1}(b)$  is not necessarily a singleton subset of  $A$ . For example, if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is given by  $f(x) = x^2$ , then the preimage of the point 4 is

$$f^{-1}(4) = \{2, -2\}.$$

A function  $f : A \rightarrow B$  is called *surjective* (or *onto*) if

$$\text{for every } b \in B \text{ there exists } a \in A \text{ such that } f(a) = b.$$

Equivalently,  $f$  is surjective if  $f(A) = B$ . This says that every element in  $B$  is “hit” by some element from  $A$ .

A function  $f : A \rightarrow B$  is called *injective* (or *one-to-one*) if

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$

Equivalently,  $f$  is injective if for all  $b \in B$ ,  $f^{-1}(b)$  contains at most one element in  $A$ .

A function  $f : A \rightarrow B$  is called *bijective* if it is both injective and surjective. Such a function sets up a *correspondence* between the elements of  $A$  and the elements of  $B$ .

**Example 7.** First we consider “real-valued functions of a real variable”. This simply means that the domain and the codomain of the function are subsets of  $\mathbb{R}$ .

- $f(x) = x^3$  is bijective;
- $g(x) = x^2$  is neither injective nor surjective;
- $h(x) = x^3 - 2x^2 - x + 2$  is surjective but not injective;
- $e(x) = 2^x$  is injective but not surjective.

Let  $A = \{-1, 1, 2\}$ . Some of the images and preimages of  $A$  are:

- $f(A) = \{-1, 1, 8\}$ ;
- $g(A) = \{1, 4\}$ ;
- $h(A) = \{0\}$ ;
- $f^{-1}(A) = \{-1, 0, \sqrt[3]{2}\}$ ;
- $g^{-1}(A) = \{-\sqrt[3]{2}, -1, 1, \sqrt[3]{2}\}$ ;
- $e^{-1}(A) = \emptyset$ .

## 8. COMPOSITION OF FUNCTIONS

Let  $A$ ,  $B$ , and  $C$  be sets and let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . The *composition* of  $f$  and  $g$  is the function

$$g \circ f : A \rightarrow C$$

given by

$$g \circ f(a) = g(f(a)).$$

The domain of  $g \circ f$  is  $A$  and the codomain is  $C$ . The range of  $g \circ f$  is the image under  $g$  of the image under  $f$  of the domain of  $f$ .

**Proposition 1.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be surjective functions. Then  $g \circ f : A \rightarrow C$  is an surjective function.*

**Proposition 2.** *Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be injective functions. Then  $g \circ f : A \rightarrow C$  is an injective function.*

**Example 8.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = x^2$  and let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $g(x) = x - 9$ . Then  $g \circ f : \mathbb{R} \rightarrow \mathbb{R}$  is given by  $g \circ f(x) = x^2 - 9$  and  $f \circ g : \mathbb{R} \rightarrow \mathbb{R}$  is given by  $f \circ g(x) = x^2 - 6x + 9$ .  $\square$

This example demonstrates that composition of functions is not a commutative operation. However, the next proposition tells us that composition of functions is associative.

**Proposition 3.** *Let  $A$ ,  $B$ ,  $C$ , and  $D$  be sets and let  $f : A \rightarrow B$ ,  $g : B \rightarrow C$ , and  $h : C \rightarrow D$  be functions. Then  $h \circ (g \circ f) = (h \circ g) \circ f$ .*

## 9. RESTRICTIONS, IDENTITIES, AND INVERSES

Let  $f : X \rightarrow Y$  be a function and let  $Z = f(X)$  be the range of  $f$ . The same function  $f$  can be viewed as a function  $f : X \rightarrow Z$ . It is standard in this case to call the function, viewed in this way, by the same name. Note that the function  $f : X \rightarrow Z$  is surjective. Thus any function is a surjective function onto its range.

Let  $f : X \rightarrow Y$  be a function and let  $A \subset X$  be a subset of the domain of  $f$ . The *restriction* of  $f$  to  $A$  is a function

$$f \upharpoonright_A : A \rightarrow Y \text{ given by } f \upharpoonright_A(a) = f(a).$$

Thus given any function and any subset of the domain, there is a function which coincides with the original one, but whose domain is the subset. For example, the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^2$  can certainly be viewed as a function on the integers, sending each integer to its square.

Let  $A$  be any set. The *identity function* on  $A$  is the function  $\text{id}_A : A \rightarrow A$  given by  $\text{id}_A(a) = a$  for every  $a \in A$ . Thus the identity function on  $A$  is that function which does nothing to  $A$ . The identity function has the property that if  $g : A \rightarrow C$ , then  $g \circ \text{id}_A = g$ , and if  $h : D \rightarrow A$ , then  $\text{id}_A \circ h = h$ .

Let  $f : A \rightarrow B$  be a function. We say that  $f$  is *invertible* if there exists a function  $g : B \rightarrow A$  such that  $g \circ f = \text{id}_A$  and  $f \circ g = \text{id}_B$ . In this case we call  $g$  the *inverse* of  $f$ . The inverse of a function  $f$  is often denoted  $f^{-1}$ .

If  $f$  is not injective, then  $f$  cannot be invertible. Sometimes we restrict the domain of  $f$  to a subset on which  $f$  is injective to invent a partial inverse.

## 10. EXERCISES

**Exercise 1.** Let  $A = \{4, 5, 6, 7, 8, 9, 10, 11\}$ ,  $B = \{2, 4, 6, 8, 10, 12, 14, 16\}$ , and  $C = \{3, 6, 9, 12, 15, 18, 21\}$ . Find the indicated set.

- (a)  $(A \cap B) \setminus C$
- (b)  $A \setminus (B \cup C)$
- (c)  $(A \setminus B) \cup C$

**Exercise 2.** Let  $A$ ,  $B$ , and  $C$  be the following subsets of  $\mathbb{N}$ :

- $A = \{n \in \mathbb{N} \mid n \leq 25\}$ ;
- $E = \{n \in A \mid n \text{ is even}\}$ ;
- $O = \{n \in A \mid n \text{ is odd}\}$ ;
- $P = \{n \in A \mid n \text{ is prime}\}$ ;
- $S = \{n \in A \mid n \text{ is a square}\}$ ;

Compute the following sets.

- (a)  $(P \cup S) \cap O$
- (b)  $(E \setminus S) \cup P$
- (c)  $(O \cap S) \times (E \cap S)$

**Exercise 3.** Let  $A = [0, 5]$ ,  $B = (2, 7)$ ,  $C = (6, 9)$ , and  $D = \{1, 3, 4, 7\}$ . Find each of the following sets.

- (a)  $(A \cup B) \setminus D$
- (b)  $B \cup (C \cap D)$
- (c)  $A \setminus D$
- (d)  $(A \cup C) \setminus D$

**Exercise 4.** Let  $A = \{x \in \mathbb{R} \mid -3 \leq x < 7\}$  and  $B = \{x \in \mathbb{R} \mid 1 < x \leq 5\}$ . Find the indicated set.

- (a)  $A$
- (b)  $B$
- (c)  $A \cup B$
- (d)  $A \cap B$
- (e)  $A \setminus B$

**Exercise 5.** Let  $A = \{1, 2, 3, 4, 5, 6\}$  and  $B = \{1, 3, 5, 7, 9, 11\}$ . Find  $C = (A \cup B) \setminus (A \cap B)$ .

**Exercise 6.** Let  $D = [2, 10]$  and  $E = (\pi, 8]$ . Find  $F = (D \setminus E) \setminus \mathbb{Z}$ .

**Exercise 7.** Sketch the graph of the set  $[1, 3] \times ([1, 4] \setminus [2, 3])$  as a subset of  $\mathbb{R}^2$ .

**Exercise 8.** Sketch the graph of the set  $([1, 5] \setminus (2, 4)) \times (\{1, 3\} \cup [4, 5])$ .

**Exercise 9.** Let  $A = [2, 3] \cup \{4\} \cup (5, 6]$ . Sketch the graph of the set  $A \times A$ .

**Exercise 10.** Sketch the graph of the set  $\{(x, y) \in \mathbb{R}^2 \mid x^2 - 6x + y^2 - 4y \leq 0\}$ .

**Exercise 11.** Draw Venn diagrams which demonstrate the following equations.

- (a)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- (b)  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- (c)  $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
- (d)  $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

**Exercise 12.** Let  $A$  and  $B$  be subsets of a set  $U$ . The *symmetric difference* of  $A$  and  $B$ , denoted  $A \triangle B$ , is the set of points in  $U$  which are in either  $A$  or  $B$  but not in both.

- (a) Draw a Venn diagram describing  $A \triangle B$ .
- (b) Find two set expressions which could be used to define  $A \triangle B$ . These expressions may use  $A$ ,  $B$ , union, intersection, complement, and parentheses,

**Exercise 13.** Find the domain of the function  $f(x) = \frac{\sqrt{x^2-3x-70}}{x^2-64}$ . Express your answer in interval notation.

**Exercise 14.** Find the range of the function  $g(x) = x^2 - 4x + 17$ . Express your answer in interval notation.

**Exercise 15.** Let  $\mathbb{N}$  be the set of natural numbers and let  $\mathbb{Z}$  be the integers. Find examples of functions  $f : \mathbb{Z} \rightarrow \mathbb{N}$  such that:

- (a)  $f$  is bijective;
- (b)  $f$  is injective but not surjective;
- (c)  $f$  is surjective but not injective;
- (d)  $f$  is neither injective nor surjective.

**Exercise 16.** Let  $\mathbb{N}$  be the set of natural numbers. Let  $A = [50, 70] \cap \mathbb{N}$ . Define a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  by  $f(n) = 3n$ . Note that  $A$  is in both the domain and the codomain of  $f$ .

- (a) Find the image  $f(A)$ .
- (b) Find the preimage  $f^{-1}(A)$ .
- (c) Is  $f$  injective? Is  $f$  surjective?

**Exercise 17.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = x^3 - 6x^2 + 11x - 3$ . Find  $f^{-1}(3)$ .

**Exercise 18.** We would like to define a function  $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q}$  by  $(p, q) \mapsto \frac{p}{q}$ . Unfortunately, this does not make sense. Fix the problem, so that the resulting function is surjective but not injective.

**Exercise 19.** We would like to define a function  $f : \mathbb{Q} \rightarrow \mathbb{Z}$  by  $\frac{p}{q} \mapsto pq$ . Unfortunately, this is not “well-defined”. Figure out what this means and fix the problem. Is the resulting function injective?

**Exercise 20.** Let  $f : X \rightarrow Y$  be a function and let  $A, B \subset X$  and  $C, D \subset Y$ . Which of the following statements are true? If the statement is false, attempt to construct a counterexample.

- (a)  $f(A \cup B) \subset f(A) \cup f(B)$
- (b)  $f(A \cup B) = f(A) \cup f(B)$
- (c)  $f(A \cap B) \subset f(A) \cap f(B)$
- (d)  $f(A \cap B) = f(A) \cap f(B)$
- (e)  $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
- (f)  $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

**Exercise 21.** Let  $f : X \rightarrow Y$  be a function. Which of the following statements are true?

- (a)  $f$  is surjective if and only if there exists  $g : Y \rightarrow X$  such that  $f \circ g = \text{id}_Y$ .
- (b)  $f$  is injective if and only if there exists  $g : Y \rightarrow X$  such that  $g \circ f = \text{id}_X$ .